MATH 2028 - Polar, Cylindrical & Spherical Coordinates

GOAL: Introduce some useful coordinate systems to evaluate double/triple integrals

So far, our consideration of multiple integrals makes use of the "Cartesian Product" structure of

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^k \times \mathbb{R}^{n-k}$$

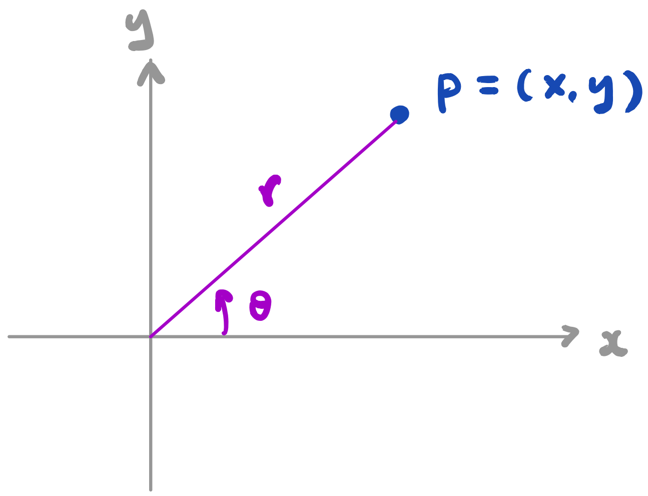
in which Cartesian/rectangular coordinates serve as a natural choice. However, there are situations where this is not the case and other coordinate system may be preferred due to the underlying symmetry.

We will focus on 3 special coordinates:

2D $\left\{ \right.$ Polar coordinates: (r, θ)

3D $\left\{ \begin{array}{l} \text{Cylindrical coordinates: } (r, \theta, z) \\ \text{Spherical coordinates: } (\rho, \phi, \theta) \end{array} \right.$

Polar coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

OR

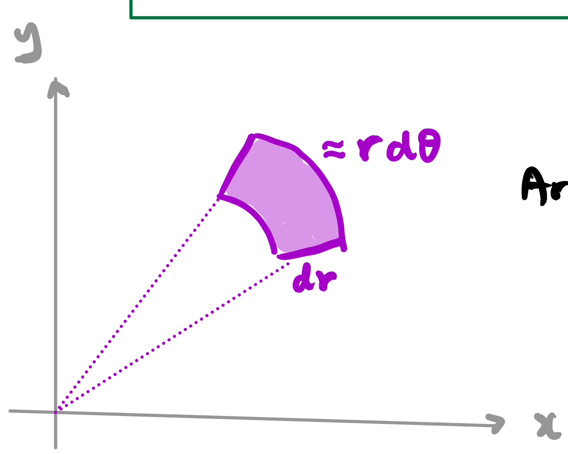
$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases}$$

To express a multiple integral in polar coord. we first rewrite a function $f(x,y)$ in (r,θ) coordinates by

$$\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta).$$

Second, we also have to deal with the distortion of an "area element":

$$\boxed{dA = dx\,dy = r\,dr\,d\theta}$$



Area of this small region $\approx r\,dr\,d\theta$

Example 1: Evaluate the double integral

$$\iint_{\Omega} \sqrt{x^2+y^2}\, dA$$

over the annulus region $\Omega = \{(x,y) \mid 1 \le x^2+y^2 \le 2\}$.

Solution: Step 1: Rewrite the function in polar coord.

$$\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$$
$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

Step 2: Rewrite the region in polar coord.

$$\Omega = \{(x,y) \mid 1 \le x^2+y^2 \le 2\}$$
$$= \{(r,\theta) \mid 1 \le r \le \sqrt{2},\ 0 \le \theta < 2\pi\}$$

Step 3: Rewrite the integral in polar coordinates.

$$\iint_{\Omega} \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_1^{\sqrt{2}} r \cdot \overbrace{r\, dr\, d\theta}^{dA}$$

$$= \int_0^{2\pi} \left[\frac{1}{3}r^3\right]_{r=1}^{r=\sqrt{2}} d\theta = \frac{1}{3}(2^{3/2}-1)\cdot 2\pi$$

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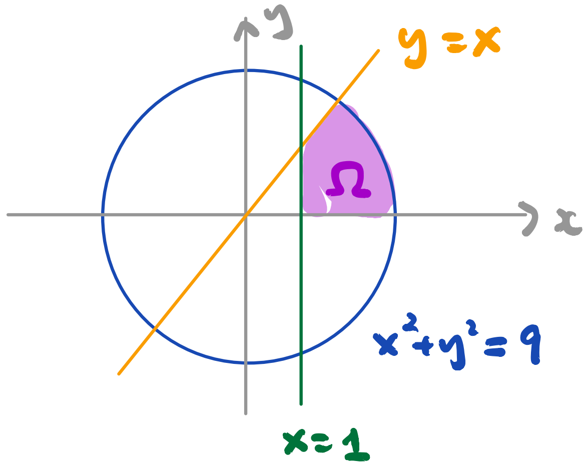
Example 2 : Evaluate the double integral

$$\iint_{\Omega} xy\, dA$$



Solution : Rewrite in polar coord.

- $f(r,\theta) = (r\cos\theta)(r\sin\theta) = r^2 \sin\theta\cos\theta$

- $\Omega = \left\{ (r,\theta) \,\middle|\, 0 \le \theta \le \frac{\pi}{4},\ \sec\theta \le r \le 3 \right\}$

$$\iint_{\Omega} xy\, dA = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{3} r^2 \sin\theta\cos\theta \cdot \overbrace{r\, dr\, d\theta}^{dA}$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4}\left(81 - \sec^4\theta\right) \sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(81 \sin\theta\cos\theta - \frac{\sin\theta}{\cos^3\theta}\right) d\theta$$

$$= \frac{1}{4}\left[\frac{81}{2}\sin^2\theta - \frac{1}{2\cos^2\theta} \right]_0^{\theta = \pi/4}$$

$$= \frac{1}{4}\left[\frac{81}{4} - 1 + \frac{1}{2} \right] = \frac{79}{16}$$

Sometimes we can use multiple integrals to help us compute certain 1D integrals. One famous example is the following:

<u>Example 3</u>: (Gaussian integral)

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

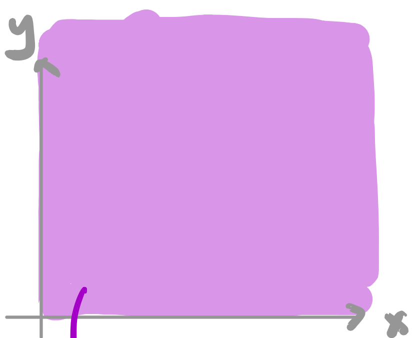
Note that this is an "improper integral".

Note that

$$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Rewriting in polar coordinates:



$\Omega = \{(x,y) \mid x, y \geq 0\}$
$= \{(r,\theta) \mid r \geq 0,\ 0 \leq \theta \leq \frac{\pi}{4}\}$

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta$$

$$= \frac{\pi}{4}.$$

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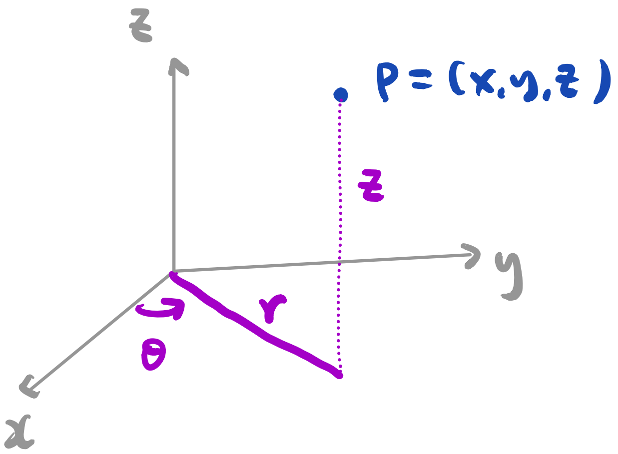
Cylindrical Coordinates



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

OR

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

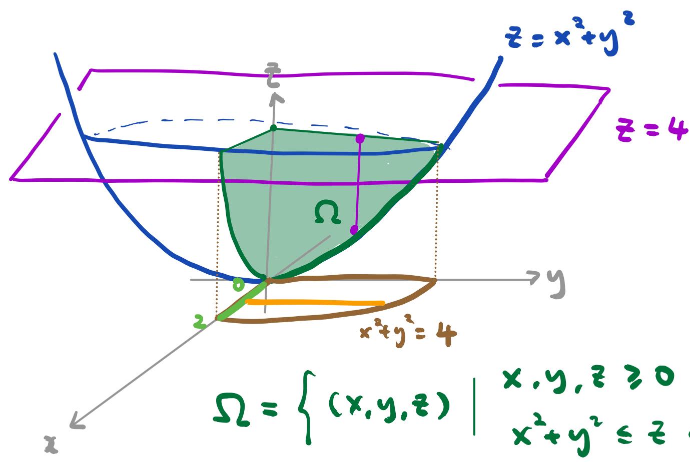
Distortion of "area element":

$$\boxed{dV = dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

Example 4: (revisited)

Evaluate the triple integral:

$$\iiint_\Omega x\,dV$$



$$\Omega = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x,y,z \geq 0 \quad \text{and} \\ x^2+y^2 \leq z \leq 4 \end{array} \right\}$$

<u>Solution</u>: Express everything in cylindrical coord.

- $\tilde{f}(r,\theta,z) = r\cos\theta$
- $\Omega = \left\{ (r,\theta,z) \;\middle|\; \begin{array}{c} 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2} \\ r^2 \le z \le 4 \end{array} \right\}$

$$\iiint_\Omega x\, dV = \int_0^{\frac{\pi}{2}} \int_0^2 \int_{r^2}^4 r\cos\theta \cdot \overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 r^2(4-r^2)\cos\theta \; dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \left[\frac{4}{3}r^3 - \frac{1}{5}r^5 \right]_{r=0}^{r=2} d\theta$$

$$= \frac{64}{15} \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$$

$$= \frac{64}{15} \left[\sin\theta \right]_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{64}{15}$$

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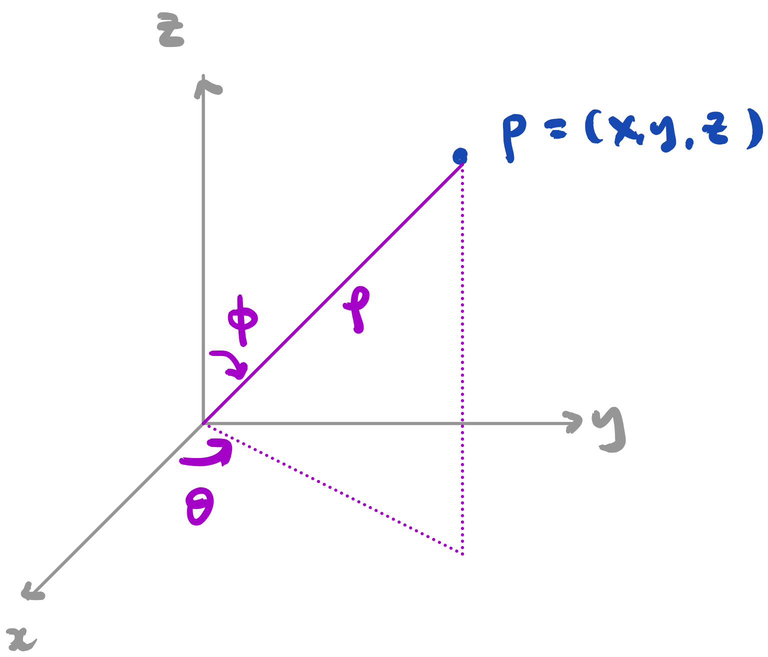
Spherical Coordinates



$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

OR

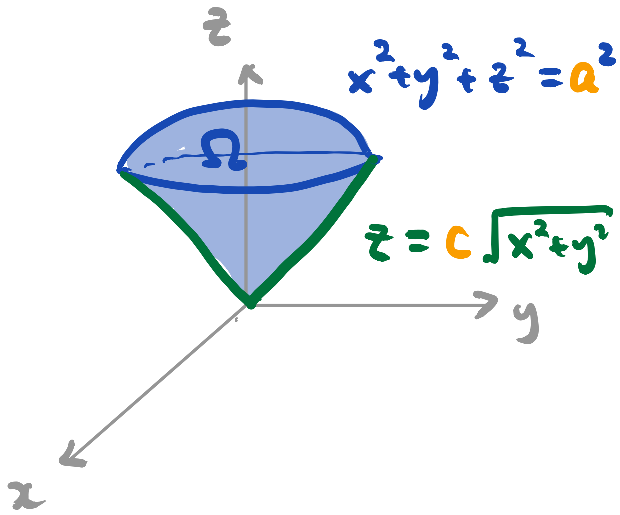
$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \phi = \cos^{-1} \dfrac{z}{(x^2+y^2+z^2)^{1/2}} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

Distortion of "area element":

$$\boxed{dV = dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

Example 5: Find the volume of the "ice-cream cone" Ω bounded by the sphere $x^2+y^2+z^2=a^2$ and the cone $z = c\sqrt{x^2+y^2}$ where $a, c > 0$ are some fixed constants.

Solution: Rewrite everything in spherical coord.



$$\Omega = \{ (\rho, \phi, \theta) \mid 0 \le \rho \le a, \quad 0 \le \phi \le \tan^{-1}\left(\frac{1}{c}\right) \quad 0 \le \theta \le 2\pi \}$$

$$\mathrm{Vol}(\Omega) = \iiint_{\Omega} 1 \, dV$$

$$= \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{c})} \int_0^{a} 1 \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$= \frac{1}{3} a^3 \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{a})} \sin\phi \, d\phi \, d\theta$$

$$= \frac{2\pi}{3} a^3 \Big[-\cos\phi \Big]_{\phi=0}^{\phi=\tan^{-1}\frac{1}{c}}$$

$$= \frac{2\pi}{3} a^3 \left(1 - \frac{c}{\sqrt{1+c^2}} \right)$$

$\blacksquare$

Spherical coordinates in $\mathbb{R}^n$

For general dimension $n \geq 3$, one can similarly introduce 1 radial coordinate ρ and $n-1$ angular coordinates $\varphi_1, \dots, \varphi_{n-1}$ s.t.

$$\begin{cases} x_1 = \rho \sin\varphi_{n-1} \sin\varphi_{n-2} \cdots \sin\varphi_2 \cos\varphi_1 \\ x_2 = \rho \sin\varphi_{n-1} \sin\varphi_{n-2} \cdots \sin\varphi_2 \sin\varphi_1 \\ \quad \vdots \\ x_{n-2} = \rho \sin\varphi_{n-1} \sin\varphi_{n-2} \cos\varphi_{n-3} \\ x_{n-1} = \rho \sin\varphi_{n-1} \cos\varphi_{n-2} \\ x_n = \rho \cos\varphi_{n-1} \end{cases}$$

Distortion of "volume element":

$$\boxed{dV = \rho^{n-1} \sin^{n-2}\varphi_{n-1} \sin^{n-3}\varphi_{n-2} \cdots\cdots \sin\varphi_2 \, d\rho \, d\varphi_1 \cdots d\varphi_{n-1}}$$

Example 6: The volume of an n-dimensional unit ball in $\mathbb{R}^n$ is equal to

$$\begin{cases} \dfrac{\pi^k}{k!} & \text{when } n = 2k \\ \dfrac{2(k!)(4\pi)^k}{(2k+1)!} & \text{when } n = 2k+1 \end{cases}$$

Ex: Prove this!